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Representations of State Space Transformations by Markov Kernels

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Every convex subset Δ of a locally convex Hausdorff space (X, τ) is equipped with the σ-algebra $\mathscr{D}$ generated by its τ-relatively open subsets. Within the set $\Omega_\sigma(\mathscr{D})$ of probability measures on $\mathscr{D}$ two particular convex subsets are considered: (a) the set $\Omega_\sigma^s(\mathscr{D})$ of probability measures with a separable support, and (b) the set $\Omega_\sigma^c(\mathscr{D})$ of probability measures with a compact convex support. If Δ is the base of a cone in X, then there exists an affine barycenter map from $\Omega_\sigma^c(\mathscr{D})$ onto Δ whose composition with the natural embedding of Δ in $\Omega_\sigma^c(\mathscr{D})$ yields the identity map on Δ, and every τ-continuous affine transformation of Δ can be represented by an affine transformation of $\Omega_\sigma^c(\mathscr{D})$ that is induced by a Markov kernel. If (X, τ) is a Banach space and Δ is a closed, bounded, generating cone base in X that is contained in a hyperplane, then analogous results are obtained with respect to $\Omega_\sigma^s(\mathscr{D})$. Since the state spaces considered in noncommutative measure theory are cone bases and every change in time of an empirical system can be thought of as an affine transformation of the associated state space (Schrödinger picture), the existence of these representation theorems implies that the time evolution of general empirical systems can be described by dynamical concepts borrowed from classical probability theory.

1. INTRODUCTION

Noncommutative measure theory deals with measures on *orthomodular posets*. It provides a frame that embraces classical measure theory, classical probability theory, and quantum probability theory. Orthomodular posets are used to describe the (logical) structure of general empirical systems, including the ones from traditional quantum mechanics. If the structure of an empirical system is represented by an orthomodular poset L, then the *states* of the system can be represented by *probability measures* on L (Beltrametti and Cassinelli, 1981; Foulis and Randall, 1972; Gudder, 1979).

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With the above interpretational context in mind, we define a *state space* as a nonempty convex set of probability measures on an orthomodular poset. Our conceptual framework suggests that the dynamical behavior of an empirical system can be described by transformations of the associated state space. The minimal requirement these transformations ought to satisfy is that they preserve mixtures (convex combinations) of states (which is sufficient for them to be affine maps). This constitutes our definition of a *state space transformation.*

Mathematically speaking, a state space Δ is a *cone base* in a real vector space X. This is the mathematical framework in which we derive our results. Instead of state space transformations, we shall thus rather speak of *cone base transformations*. Choosing this more general setting, we can also accommodate the convexity approach to quantum mechanics taken by Ludwig (1983/1985) and Mielnik (1974).

In order to represent cone base transformations by Markov kernels, we assume that X is equipped with a locally convex Hausdorff topology τ and consider probability measures on the σ-algebra $\mathscr{D}$ of Borel subsets of Δ. From a Bayesian point of view such probability measures can be interpreted as credibilities, if Δ is a state space (Randall and Foulis, 1976). By $\Omega_\sigma(\mathscr{D})$ we denote the set of all probability measures on $\mathscr{D}$, while $\Omega_\sigma^s(\mathscr{D})$, resp. $\Omega_\sigma^c(\mathscr{D})$, stands for the set of probability measures with a separable, resp. compact convex, support. Every continuous cone base transformation T can be assigned a Markov kernel P_T on $\Delta \times \mathscr{D}$ as follows:

$$P_T(x, D) = 1_D(T(x)), \qquad x \in \Delta, \quad D \in \mathscr{D}$$

where $1_D(x)$ denotes the indicator function of the set D. The kernel P_T induces an affine transformation $\tilde{T}$ of $\Omega_\sigma(\mathscr{D})$ that maps $\Omega_\sigma^s(\mathscr{D})$ and $\Omega_\sigma^c(\mathscr{D})$ into themselves.

For a complete representation we need to establish some sort of an affine correspondence between the set Δ and a suitable convex subset of $\Omega_\sigma(\mathscr{D})$. The easiest way to do this is by means of the map

$$x \in \Delta \overset{\delta}{\mapsto} P_{id_\Delta}(x, \cdot\,)$$

which embeds Δ in each of the sets $\Omega_\sigma(\mathscr{D})$, $\Omega_\sigma^s(\mathscr{D})$, and $\Omega_\sigma^c(\mathscr{D})$. This map is affine in the sense that

$$\int_\Delta f(z)\delta\left(\sum t_i x_i\right)(dz) = \sum t_i \int_\Delta f(z)\delta(x_i)\,(dz)$$

holds for all continuous linear functionals f on X, all $x_1, \ldots, x_n \in \Delta$, and all $t_1, \ldots, t_n \in \mathbf{R}$ with $\sum t_i = 1$ and $\sum t_i x_i \in \Delta$. The canonical way of trying to map $\Omega_\sigma(\mathscr{D})$ into Δ is by assigning each $\omega \in \Omega_\sigma(\mathscr{D})$ its barycenter functional $j(\omega)$

in $X^{*\prime}$ (the algebraic dual space of X^*). This functional is uniquely defined by the condition

$$j(\omega)(f) = \int_\Delta f(z)\, \omega(dz)$$

for all continuous linear functionals f on X. If $j(\omega)$ belongs to Δ (considered as a subset of $X^{*\prime}$), then ω is said to have a *barycenter* x_ω in Δ.

We show that: (a) every $\omega \in \Omega_\sigma^c(\mathscr{D})$ has a barycenter in Δ; and (b) if (X, τ) is a Banach space that is generated by a closed, bounded cone base Δ, contained in a hyperplane, then every $\omega \in \Omega_\sigma^s(\mathscr{D})$ has a barycenter in Δ.

Moreover, we show that in each of these cases

$$T(x_\omega) = x_{\tilde{T}(\omega)} \tag{$*$}$$

holds for all continuous cone base transformations T of Δ and all $\omega \in \Omega_\sigma^c(\mathscr{D})$ [case (a)], resp. $\omega \in \Omega_\sigma^s(\mathscr{D})$ [case (b)].

We apply these results to the state spaces of orthomodular posets and their transformations. By $\Omega(L)$ we denote the set of states on an orthomodular poset L. This set is compact in the product topology τ of $\mathbf{R}^L$. Hence, if $\Delta = \Omega(L)$, then every $\omega \in \Omega_\sigma(\mathscr{D})$ has a compact support and thus a barycenter x_ω in Δ and satisfies equation ($*$). Although smaller state spaces Δ are usually not compact in this topology, it is often possible to find a norm on $\mathrm{lin}(\Delta)$ such that X, the topology τ induced by this norm, and Δ fit into setting (b). For instance, this is true of the two state spaces formed by the σ-additive, resp. completely additive, elements of $\Omega(L)$. In all examples associated with traditional Hilbert space quantum mechanics, the state spaces Δ are even separable in their respective norm-topologies, so that every $\omega \in \Omega_\sigma(\mathscr{D})$ has a barycenter in Δ and satisfies equation ($*$) with respect to any state space transformation T. Every such example is derived from a separable complex Hilbert space H, by taking the lattice of orthogonal projections on H as the orthomodular poset, the convex set of von Neumann density operators as the state space Δ, and the trace-norm topology on Δ as τ.

Although the main concern of this paper is to show that the time evolution of general empirical systems can be described in terms of dynamical concepts borrowed from classical probability theory, some auxiliary results might be of interest in their own right. We think in particular of the results concerning barycenters of measures. Whereas boundary measures on compact convex sets have been studied intensely (Alfsen, 1971), it seems that less attention has been paid to more general measures on noncompact convex sets. Our approach might be a small contribution to filling this gap.

2. PRELIMINARIES

Let $(L, \leq, ')$ be an *orthocomplemented poset*, $\#L > 1$, and let the smallest and largest elements of L be denoted by 0 and 1, respectively. A pair (p, q) of elements of L is said to be *orthogonal*, denoted $p \perp q$, if $p \leq q'$. Notice that the relation $\perp$ is symmetrical. A subset A of L is said to be *orthogonal* if $p \perp q$ holds for all $p, q \in L$ with $p \neq q$.

An *orthomodular poset* (*OMP*) is an orthocomplemented poset $(L, \leq, ')$ satisfying: (i) if $p \perp q$, then $p \vee q$ exists; and (ii) if $p \leq q$, then $q = p \vee (q \wedge p')$ (orthomodular identity).

An orthomodular poset $(L, \leq, ')$, for which $(L, \leq)$ is a lattice is called an *orthomodular lattice* (*OML*). For more details see, e.g., Gudder (1979), Kalmbach (1983), and Rüttimann (1979).

We mention two important examples of orthomodular lattices:

1. Let $(L, \leq)$ be a Boolean lattice, and let $': L \to L$ be its uniquely defined complementation map. Then $(L, \leq, ')$ is an *OML*.

2. Let H be a complex Hilbert space, and let $(\mathscr{L}, \subseteq)$ be the lattice formed by the closed subspaces of H. Then the triple $(\mathscr{L}, \subseteq, ^{\perp})$, where $^{\perp}$ denotes the map assigning each $L \in \mathscr{L}$ its unique orthogonal complement, is an *OML*.

Let $(L_1, \leq_1, '^1)$ and $(L_2, \leq_2, '^2)$ be two orthomodular posets. A map $\Phi: L_1 \to L_2$ satisfying (i) $\Phi(1_1) = 1_2$ and (ii) if $p \perp_1 q$, then $\Phi(p) \perp_2 \Phi(q)$ and $\Phi(p \vee_1 q) = \Phi(p) \vee_2 \Phi(q)$, is called a *homomorphism* from $(L_1, \leq_1, '^1)$ to $(L_2, \leq_2, '^2)$. Notice that, as a consequence of orthomodularity, a homomorphism Φ preserves order and orthocomplementation as well. A homomorphism $\Phi: L_1 \to L_2$ is called a *c-homomorphism*, resp. *σ-homomorphism*, if for all orthogonal, resp. countable orthogonal, subsets C of L_1 with supremum in $(L_1, \leq_1)$, $\sup_2 \Phi(C)$ exists in $(L_2, \leq_2)$ and equals $\Phi(\sup_1 C)$. Consequently, a c-homomorphism, resp. σ-homomorphism, from $(L_1, \leq_1, '^1)$ into $(L_2, \leq_2, '^2)$ maps maximal orthogonal, resp. countable maximal orthogonal, subsets of L_1 into maximal orthogonal subsets of L_2 (which may contain 0_2).

If $(L, \leq, ')$ is an orthomodular poset, then an element μ of the vector space $\mathbf{R}^L$ is said to be a *measure* on L if $p \perp q$ implies $\mu(p \vee q) = \mu(p) + \mu(q)$. The measures on L form a linear subspace of $\mathbf{R}^L$. A measure μ is said to be *positive* if $\mu(p) \geq 0$ for all $p \in L$. By $K(L)$ we denote the collection of positive measures on L. Notice that the map $L \to 0$ is an element of $K(L)$. An element μ of $K(L)$ with $\mu(1) = 1$ is said to be a *state* or *probability measure* on L. By $\Omega(L)$ we denote the (convex) set of states. The linear space generated by $\Omega(L)$ is denoted by $J(L)$. Its elements are called *Jordan measures*. Since $\Omega(L)$ is convex, $K(L)$ equals $\mathbf{R}_+\Omega(L) = \{t\mu : \mu \in \Omega(L), t \in \mathbf{R}_+\}$ and $J(L)$ equals $K(L) - K(L) = \{\sigma - \rho : \rho, \sigma \in K(L)\}$.

A measure μ on L is said to be *completely additive*, resp. *σ-additive*, if for every maximal, resp. countable maximal, orthogonal subset A of L

$$\mu(1) = \lim(\mu(\sup B))_{B \in A^f},$$

where $(A^f, \subseteq)$ denotes the collection of finite subsets of A directed by set inclusion. By $\Omega_c(L)$, resp. $\Omega_\sigma(L)$, we denote the convex set formed by the completely additive, resp. σ-additive, states on L. For a more detailed treatment of these measure-theoretic concepts we refer to Fischer and Rüttimann (1978) and Rüttimann (1979, 1985).

3. MEASURES ON CONVEX SETS

Let X be a real vector space. A subset C of X is said to be *convex*, resp. *affine*, if for all $x, y \in C$ and $t \in [0, 1]$, resp. $t \in \mathbf{R}$, $tx + (1-t)y$ belongs to C. A subset C of X is said to be *positive* if it contains 0 and $sx + ty$ belongs to C, for all $x, y \in C$ and $s, t \in \mathbf{R}_+$. A positive subset C with $C \cap -C = \{0\}$ is called a *cone*. For every subset C of X we denote its convex, resp. affine, resp. positive, hull by $\mathrm{con}(C)$, resp. $\mathrm{aff}(C)$, resp. $\mathrm{pos}(C)$. If C is nonempty and convex then $\mathrm{pos}(C)$ equals $\mathbf{R}_+ C = \{tx : x \in C, t \in \mathbf{R}_+\}$. A convex subset C of X is said to be a *cone base* if $0 \notin \mathrm{aff}(C)$. Indeed, the positive hull of a convex set C is a cone if and only if C is a cone base.

For every subset C of X let $\mathrm{acon}(C) = \mathrm{con}(C \cup -C)$ (*absolute convex hull* of C). If C is convex, then $\mathrm{acon}(C)$ coincides with the set $\{s\sigma - r\rho : \rho, \sigma \in C;\ r, s \in \mathbf{R}_+, r + s = 1\}$. If C is a nonempty convex subset of X that generates X [i.e., $X = \mathrm{lin}(C)$], then we may define the *Minkowski functional* $\rho_{\mathrm{acon}(C)}(x) = \inf\{t \in \mathbf{R}_+ : x \in t\, \mathrm{acon}(C)\}$, which is a seminorm on X.

In the sequel we further assume that the space X carries a locally convex Hausdorff topology τ. By X^* we denote the topological dual space of (X, τ), and $X^{*\prime}$ will stand for the algebraic dual space of X^*. We shall use the symbol J_X to denote the canonical embedding map from X into $X^{*\prime}$.

From now on let Δ be a nonempty convex subset of X, and let τ_Δ denote the relative topology of τ on Δ. If Δ is τ-bounded (i.e., for each open neighborhood U of 0 there exists $t > 0$ such that $t\Delta \subseteq U$), then every $f \in X^*$ maps Δ into a bounded subset of $\mathbf{R}$. By $A_c(\Delta)$ we denote the vector space of τ_Δ-continuous affine functionals on Δ [i.e., a map $f: \Delta \to \mathbf{R}$ belongs to $A_c(\Delta)$ if and only if it is τ_Δ-continuous and

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y)$$

for all $x, y \in \Delta$ and all $t \in \mathbf{R}$ with $tx + (1-t)y \in \Delta$]. For each $f \in X^*$ the restriction $f|\Delta$ belongs to $A_c(\Delta)$.

Let $\mathscr{D}$ denote the σ-algebra on Δ generated by the τ_Δ-open subsets of Δ, and let $\Omega_\sigma(\mathscr{D})$ stand for the convex set of σ-additive probability measures on $\mathscr{D}$. A standard argument shows that for every $f \in X^*$ the restriction $f|\Delta$ is $\mathscr{D}$-measurable. For every $x \in \Delta$ and each $D \in \mathscr{D}$ we define

$$\delta(x)(D) = \begin{cases} 1 & \text{if} \quad x \in D \\ 0 & \text{if} \quad x \notin D \end{cases} \tag{1}$$

Notice that $\delta(x)(D) = 1_D(x)$, where $1_D(x)$ denotes the indicator function of the set D. Clearly, $\delta(x)$ belongs to $\Omega_\sigma(\mathscr{D})$, for every $x \in \Delta$. Moreover, since the topology τ is Hausdorff, the map $\delta: \Delta \to \Omega_\sigma(\mathscr{D})$ is injective.

For $\omega \in \Omega_\sigma(\mathscr{D})$ and $f \in X^*$ we define

$$j(\omega)(f) = \int_\Delta f(x)\, \omega(dx) \tag{2}$$

provided the integral exists. If Δ is τ-bounded, then (i) $j(\omega)(f)$ is defined for all $\omega \in \Omega_\sigma(\mathscr{D})$ and $f \in X^*$, (ii) $j(\omega)$ is a linear functional on X^*, for every $\omega \in \Omega_\sigma(\mathscr{D})$, and (iii) $j: \Omega_\sigma(\mathscr{D}) \to X^{*\prime}$ is an affine map. It is easy to prove that, for each $x \in \Delta$, $j(\delta(x))$ is defined as an element of $X^{*\prime}$ and

$$J_X(x) = j(\delta(x)) \tag{3}$$

If $\omega \in \Omega_\sigma(\mathscr{D})$ is such that $j(\omega)$ is defined as an element of $X^{*\prime}$ which also belongs to $J_X(\Delta)$, then we set

$$j'(\omega) = x_\omega \tag{4}$$

where x_ω is the unique element of Δ satisfying $j(\omega) = J_X(x_\omega)$. The element x_ω is called the *barycenter* of ω in Δ (Alfsen, 1971).

We now define two subsets of $\Omega_\sigma(\mathscr{D})$ that will be important in the sequel.

Definition 3.1. An element ω of $\Omega_\sigma(\mathscr{D})$ is said to have a *separable support* if there exists a τ-separable set D in $\mathscr{D}$ such that $\omega(D) = 1$. The collection of all such elements will be denoted by $\Omega_\sigma^s(\mathscr{D})$.

An element ω of $\Omega_\sigma(\mathscr{D})$ is said to have a *compact convex support* if there exists a τ-compact convex subset D of Δ such that $\omega(D) = 1$. The collection of all such elements will be denoted by $\Omega_\sigma^c(\mathscr{D})$.

It is easy to see that $\Omega_\sigma^s(\mathscr{D})$ is a convex subset of $\Omega_\sigma(\mathscr{D})$. To see that the set $\Omega_\sigma^c(\mathscr{D})$ is convex, observe that the convex hull of the union of two compact convex sets is again compact (Koethe, 1969). Obviously, for each $x \in \Delta$ the measure $\delta(x)$ belongs to both $\Omega_\sigma^s(\mathscr{D})$ and $\Omega_\sigma^c(\mathscr{D})$.

Theorem 3.2. If Δ is a nonempty convex subset of X, then for each $\omega\in\Omega_\sigma^c(\mathscr{D})$ there exists $x_\omega\in\Delta$ such that

$$\int_\Delta f(x)\,\omega(dx)=f(x_\omega)$$

for all $f\in A_c(\Delta)$.

Proof. Let ω be an arbitrary element of $\Omega_\sigma^c(\mathscr{D})$. Then there exists a compact convex subset Γ of Δ such that $\omega(\Gamma)=1$. Let $f\in A_c(\Delta)$ and $n\in\mathbf{N}$ be arbitrary. Since Γ is compact, there exists $K\in\mathbf{N}$ such that $f(\Gamma)\subseteq(-K, K)$. We now define a partition of Γ as follows:

$$P(f,n)=\left\{f^{-1}\left(\left[-K+\frac{j}{n}, -K+\frac{j+1}{n}\right)\right)\cap\Gamma : j=0,\ldots,2Kn-1\right\}$$

Clearly, all elements D of $P(f,n)$ belong to $\mathscr{D}$ and satisfy $d(f(D))<1/n$, where d denotes the diameter function on $2^{\mathbf{R}}$. For every finite subset $F=\{f_1,\ldots,f_k\}$ of $A_c(\Delta)$ and each $n\in\mathbf{N}$, let $P(F,n)$ denote the coarsest refinement of the partitions $P(f_1,n),\ldots,P(f_k,n)$, i.e.,

$$P(F,n)=\left\{\bigcap_{i=1}^{k} D_i : D_i\in P(f_i,n),\, i=1,\ldots,k\right\}-\{\varnothing\}$$

let $h_{F,n}\colon P(F,n)\to\Gamma$ be a map that satisfies $h_{F,n}(D)\in D$ for all $D\in P(F,n)$, and let

$$x_{\omega;F,n}=\sum_{D\in P(F,n)} h_{F,n}(D)\omega(D)$$

On the collection Ξ of all such pairs (F,n) we introduce a partial order $\preceq$ as follows:

$$(F_1,n_1)\preceq(F_2,n_2)\Leftrightarrow F_1\subseteq F_2 \quad\text{and}\quad n_1\leq n_2$$

Obviously, $(\Xi,\preceq)$ is a directed set, so that $(x_{\omega;F,n})_{(F,n)\in\Xi}$ becomes a net. Since Γ is compact, there exists an element x_ω in Γ and a directed subset $(\Xi',\preceq)$ of $(\Xi,\preceq)$ such that

$$x_\omega=\tau_\Delta-\lim(x_{\omega;\xi})_{\xi\in\Xi'}$$

We claim that $j(\omega)(f)=f(x_\omega)$ for all $f\in A_c(\Delta)$:

Let $f\in A_c(\Delta)$ and $\varepsilon>0$ be arbitrary. Then there exists $\xi_0\in\Xi'$ such that $|f(x_{\omega;\xi})-f(x_\omega)|<\varepsilon/2$ holds for all $\xi\in\Xi'$ with $\xi\succeq\xi_0$. Moreover, there exists

$(F, n) \in \Xi'$ such that $(F, n) \succeq \xi_0$, $f \in F$, and $1/n < \varepsilon/2$. We thus have

$$|j(\omega)(f) - f(x_{\omega;F,n})| = |j(\omega)(f) - \sum_{D \in P(F,n)} f(h_{F,n}(D))\omega(D)|$$
$$\leq \omega(\Gamma)\frac{1}{n}$$
$$< \frac{\varepsilon}{2}$$

Hence $|j(\omega)(f) - f(x_\omega)| < \varepsilon$. This concludes the proof. ■

Corollary 3.3. If Δ is a nonempty, τ-compact convex subset of X, then the domain of definition of the map j' is $\Omega_\sigma(\mathscr{D})$.

Proof. If Δ is τ-compact, then $\Omega_\sigma^c(\mathscr{D})$ coincides with $\Omega_\sigma(\mathscr{D})$. The assertion then follows from Theorem 3.2, observing that for each $f \in X^*$ the restriction $f|\Delta$ belongs to $A_c(\Delta)$. ■

In many applications the topology τ will be induced by a norm $\|\cdot\|$ on X. A sufficient condition for all elements of $\Omega_\sigma^s(\mathscr{D})$ to have a barycenter in Δ is then given by the following theorem.

Theorem 3.4. If X is a Banach space and Δ is a nonempty, closed, and bounded convex subset of X, then $\Omega_\sigma^s(\mathscr{D})$ is in the domain of definition of j'.

Proof. We shall use the results on the Bochner integral listed in the Appendix. In the present context, Δ, $\mathscr{D}$, and an arbitrary element ω of $\Omega_\sigma^s(\mathscr{D})$ take the roles of S, Σ, and μ, respectively. We set $s = \sup\{\|x\| : x \in \Delta\}$ and observe that the conditions of Lemma A.1 and Theorem A.2 are met by taking $X = \Delta$, $f = \mathrm{id}_\Delta$, $Y = \mathbf{R}$, and $g = s$. Hence, the map id_Δ is ω-integrable and we can define

$$x_\omega = \int_\Delta x\, \omega(dx)$$

Now, let L be an arbitrary element of X^*. Then $L \circ f = L \circ \mathrm{id}_\Delta$ is ω-measurable. Since $L(\Delta)$ is bounded, $L \circ f$ is ω-integrable. It then follows from Theorem A.3 that

$$L \int_\Delta x\, \omega(dx) = \int_\Delta L(x)\, \omega(dx)$$

Assume that x_ω does not belong to Δ. Then, since Δ is closed and convex, there exists $h \in X^*$ and $t \in \mathbf{R}$ such that

$$h(\Delta) \leq t < h(x_\omega)$$

[Koethe (1969), Separation Theorem]. On the other hand, we have

$$h(x_\omega) = \int_\Delta h(x)\, \omega(dx) \leq t \int_\Delta \omega(dx) = t$$

Since this is a contradiction, we conclude that $x_\omega \in \Delta$. ■

Corollary 3.5. If X is a Banach space and Δ is a closed, bounded, and separable convex subset of X, then the domain of definition of the map j' is $\Omega_\sigma(\mathscr{D})$.

Proof. If Δ is separable, then $\Omega_\sigma^s(\mathscr{D})$ coincides with $\Omega_\sigma(\mathscr{D})$. ■

4. CONE BASE TRANSFORMATIONS

As in Section 3, let (X, τ) be a Hausdorff locally convex space. However, we now assume that Δ is a nonempty cone base in X that generates X [i.e., $X = \text{lin}(\Delta)$]. Again, let $\mathscr{D}$ denote the σ-algebra on Δ generated by the τ_Δ-open subsets of Δ.

Definition 4.1. A map T: $\Delta \to \Delta$ is said to be a *cone base transformation* (*CBT*) of Δ if it preserves convex combinations {i.e., if $T(tx + (1-t)y) = tT(x) + (1-t)T(y)$ for all $x, y \in \Delta$ and $t \in [0, 1]$}.

Theorem 4.2. Every *CBT* T: $\Delta \to \Delta$ has a unique extension to a linear transformation T' of X.

Proof. We only sketch the proof [for a more detailed argument see Gudder (1977), Theorem 2]. Since Δ is a cone base in X, every element y of $\text{pos}(\Delta) - \{0\}$ determines a unique pair $(x(y), t(y)) \in \Delta \times \mathbf{R}_+$ such that $y = t(y)x(y)$. One then shows that the map T'': $\text{pos}(\Delta) \to \text{pos}(\Delta)$, defined by

$$T''(y) = \begin{cases} t(y)T(x(y)) & \text{if} \quad y \neq 0 \\ 0 & \text{if} \quad y = 0 \end{cases}$$

is positive [i.e., $T''(sy + tz) = sT''(y) + tT''(z)$ for all $y, z \in \text{pos}(\Delta)$ and $s, t \in \mathbf{R}_+$]. Since Δ is convex and generates X, every element z of X can be represented in the form $z = y - x$, with x and y being suitable elements of $\text{pos}(\Delta)$. After setting $T'(z) = T''(y) - T''(x)$, one shows that this definition does not depend on the particular choice of the representing elements x and y, and that the map T': $X \to X$ is indeed linear. Since Δ generates X, this extension is unique. ■

Definition 4.3. A *CBT* $T: \Delta \to \Delta$ is said to be *continuous* if it is a continuous map from the topological space (Δ, τ_Δ) into itself.

Since the σ-algebra $\mathscr{D}$ is generated by the τ_Δ-open subsets of Δ, all continuous *CBT*s of Δ are $\mathscr{D}$-$\mathscr{D}$-measurable.

Let T be a *CBT* of Δ. For all $x \in \Delta$, $D \in \mathscr{D}$, and $\omega \in \Omega_\sigma(\mathscr{D})$, we define (Alfsen, 1971)

$$P_T(x, D) = \delta(Tx)(D) \tag{5}$$

and

$$\tilde{T}(\omega)(D) = \omega(T^{-1}(D)) \tag{6}$$

Lemma 4.4. Assume that T is a continuous *CBT* of Δ. Then, for every $\omega \in \Omega_\sigma(\mathscr{D})$, $\tilde{T}(\omega)(\cdot)$ is again an element of $\Omega_\sigma(\mathscr{D})$. More precisely, $\tilde{T}$ maps each of the sets $\Omega_\sigma(\mathscr{D})$, $\Omega_\sigma^s(\mathscr{D})$, and $\Omega_\sigma^c(\mathscr{D})$ affinely into itself.

Proof. The first statement is obvious. That the map $\tilde{T}$ is affine on $\Omega_\sigma(\mathscr{D})$ is straightforward, too. Let ω be an arbitrary element of $\Omega_\sigma^s(\mathscr{D})$. Then there exists a separable set D in $\mathscr{D}$ with $\omega(D) = 1$. Since T is continuous, the set $T(D)$ is also separable and we have $\tilde{T}(\omega)(T(D)) = \omega(T^{-1}(T(D)) \geq \omega(D) = 1$. Hence $\tilde{T}$ maps $\Omega_\sigma^s(\mathscr{D})$ affinely into itself. The last statement is proved similarly, using the fact that the image of a compact convex set under a continuous affine map is again compact and convex (Koethe, 1969). ■

Lemma 4.5. Assume that T is a continuous *CBT* of Δ. Then the map $P_T: \Delta \times \mathscr{D} \to \mathbf{R}$ is a Markov kernel, i.e., (i) for each $x \in \Delta$, $P_T(x, \cdot)$ is an element of $\Omega_\sigma(\mathscr{D})$; and (ii) for each $D \in \mathscr{D}$, $P_T(\cdot, D)$ is $\mathscr{D}$-measurable.

Proof. Part (i) is obvious from the definition of P_T. (ii) Let D be an arbitrary element of $\mathscr{D}$, and let B be an arbitrary Borel set in $\mathbf{R}$. We then have

$$\begin{aligned} P_T(\cdot, D)^{-1}(B) &= \{x \in \Delta : P_T(x, D) \in B\} \\ &= \{x \in \Delta : \delta(Tx)(D) \in B\} \\ &= \{x \in \Delta : 1_D(Tx) \in B\} \\ &= \begin{cases} \varnothing & \text{if } B \cap \{0, 1\} = \varnothing \\ \Delta & \text{if } \{0, 1\} \subseteq B \\ T^{-1}(D) & \text{if } 1 \in B, 0 \notin B \\ T^{-1}(D^c) & \text{if } 0 \in B, 1 \notin B \end{cases} \end{aligned}$$

Since T is $\mathscr{D}$-$\mathscr{D}$-measurable, $P_T(\cdot, D)^{-1}(B) \in \mathscr{D}$. ■

The Markov kernel P_T associated with a continuous *CBT* of Δ defines an affine transformation of $\Omega_\sigma(\mathscr{D})$ as follows (Bauer, 1978):

$$\tilde{P}_T(\omega) = \int_\Delta \omega(dx)\, P(x, \cdot) \tag{7}$$

Lemma 4.6. Assume that T is a continuous *CBT* of Δ. Then the two affine transformations $\tilde{P}_T$ and $\tilde{T}$ of $\Omega_\sigma(\mathscr{D})$ coincide.

Proof. Let ω and D be arbitrary elements of $\Omega_\sigma(\mathscr{D})$ and $\mathscr{D}$, respectively. We then have

$$\begin{aligned}\tilde{P}_T(\omega)(D) &= \int_\Delta P_T(x, D)\, \omega(dx) \\ &= \int_\Delta 1_D(Tx)\, \omega(dx) \\ &= \omega(T^{-1}(D)) \\ &= \tilde{T}(\omega)(D) \quad \blacksquare\end{aligned}$$

In view of Lemma 4.4 we allow ourselves to also write $\tilde{T}$ and $\tilde{P}_T$ when these maps are considered on the restricted domains $\Omega^s_\sigma(\mathscr{D})$ and $\Omega^c_\sigma(\mathscr{D})$.

Theorem 4.7. For every *CBT* T of Δ the diagrams of (a) Figure 1 and (b) Figure 2 are commutative.

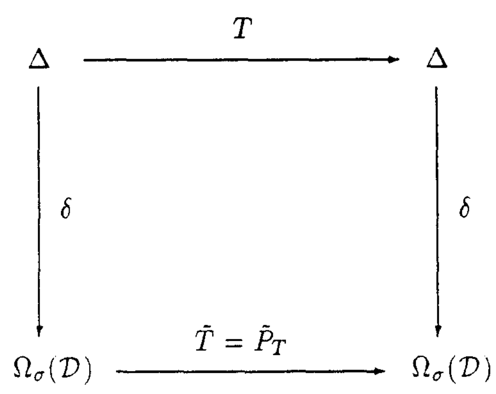

Fig. 1

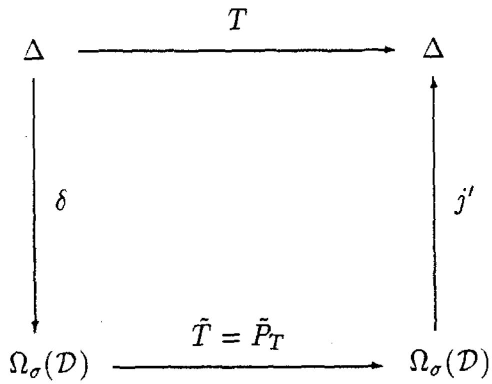

Fig. 2

Proof. (a) Let x and D be arbitrary elements of Δ and $\mathscr{D}$, respectively. Then we have

$$\begin{aligned}\tilde{T}(\delta(x))(D) &= \delta(x)(T^{-1}(D)) \\ &= 1_{T^{-1}(D)}(x) \\ &= 1_D(Tx) \\ &= \delta(Tx)(D)\end{aligned}$$

(b) Let x be an arbitrary element of Δ. Then, by (a), $\tilde{T}(\delta(x)) = \delta(Tx)$. Since $j' \circ \delta = \mathrm{id}_\Delta$, we obtain

$$\begin{aligned}T(x) &= j' \circ \delta \circ T(x) \\ &= j' \circ \tilde{T} \circ \delta(x) \quad \blacksquare\end{aligned}$$

Theorem 4.8. For every continuous *CBT* T of Δ the diagram of Figure 3 is commutative.

Proof. Let ω and f be arbitrary elements of $\Omega_\sigma^c(\mathscr{D})$ and X^*, respectively. Then there exists a τ-compact convex subset D of $\mathscr{D}$ such that $\omega(D) = 1$. Since, by Theorem 4.2, the map $T: \Delta \to \Delta$ is affine, $f \circ T$ belongs to $A_c(\Delta)$. Thus, by Theorem 3.2,

$$f \circ T(j'(\omega)) = \int_\Delta f \circ T(x)\, \omega(dx)$$

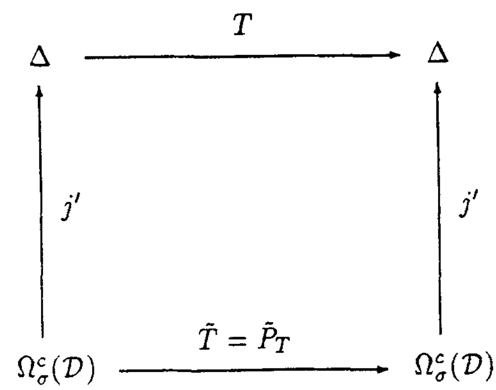

Fig. 3

Let $K \in \mathbf{N}$ be such that $|f(x)| < K$, for all $x \in D$. Then

$$\int_\Delta f \circ T(x)\, \omega(dx) = \lim_{n\to\infty} \sum_{l=0}^{2Kn-1} \left(-K+\frac{l}{n}\right) \times \omega\left((f\circ T)^{-1}\left(\left[-K+\frac{l}{n}, -K+\frac{l+1}{n}\right)\right)\right)$$

$$= \lim_{n\to\infty} \sum_{l=0}^{2Kn-1} \left(-K+\frac{l}{n}\right) \times \omega\left(T^{-1}\left(f^{-1}\left(\left[-K+\frac{l}{n}, -K+\frac{l+1}{n}\right)\right)\right)\right)$$

$$= \int_\Delta f(x)\, (\tilde{T}\omega)(dx)$$

$$= j(\tilde{T}\omega)(f)$$

$$= f(j' \circ \tilde{T}(\omega))$$

This shows that $T \circ j'(\omega) = j' \circ \tilde{T}(\omega)$, for all $\omega \in \Omega^c_\sigma(\mathscr{D})$. ■

Corollary 4.9. If Δ is τ-compact and $T\colon \Delta \to \Delta$ is a continuous *CBT* of Δ, then the diagram of Figure 4 is commutative.

The following theorem pertains to the case when the topology τ is induced by a norm and the probability measures of interest are those with a separable support. Its proof is postponed to the end of the section, since

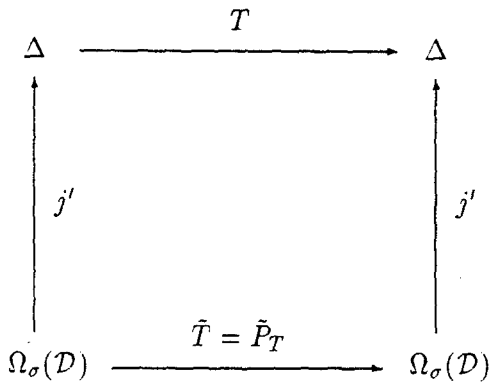

Fig. 4

it requires a few preliminary considerations about the continuity properties of the linear extension T' of a *CBT* T.

Theorem 4.10. Assume that the topology τ is induced by a norm $\|\cdot\|$ and that $(X, \|\cdot\|)$ is a Banach space. If Δ is closed and bounded and there exists $f \in X^*$ such that $f(\Delta) = \{1\}$, then the diagram of Figure 5 is commutative for every *CBT* $T: \Delta \to \Delta$.

Corollary 4.11. Assume that the topology τ is induced by a norm $\|\cdot\|$ and that $(X, \|\cdot\|)$ is a separable Banach space. If Δ satisfies the same conditions as in the preceding theorem, then the diagram of Figure 4 is commutative for every *CBT* $T: \Delta \to \Delta$.

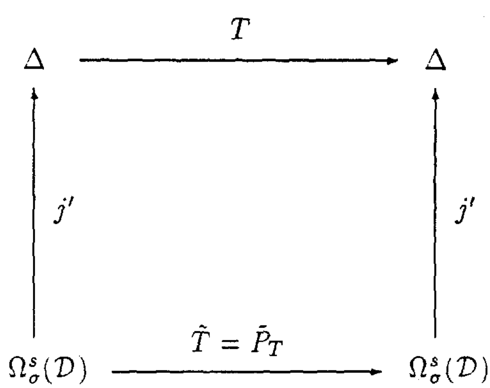

Fig. 5

The following results are needed for the proof of Theorem 4.10.

Lemma 4.12. Let X be a normed space, and let Δ be a closed, bounded, convex subset of X such that $f(\Delta)=\{1\}$, for some $f\in X^*$. Then: (i) the set Δ is a cone base in X; and (ii) the cone $\text{pos}(\Delta)$ is closed.

Proof. From $f(\Delta)=\{1\}$ it follows that $f(\text{aff}(\Delta))=\{1\}$. Hence $0\notin\text{aff}(\Delta)$. This proves (i). (ii) Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in $\text{pos}(\Delta)$ converging to an element x of X. Then there exist sequences $(y_n)_{n\in\mathbf{N}}$ in Δ and $(t_n)_{n\in\mathbf{N}}$ in $\mathbf{R}_+$ such that $x_n=t_n y_n$. Since $f(x)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}t_n f(y_n)$, and $f(\Delta)=\{1\}$, it follows that $t_n\to f(x)$. Hence $f(x)\geq 0$. If $f(x)=0$, then $(x_n)_{n\in\mathbf{N}}$ converges to 0 in norm, since $\|x_n\|=t_n\|y_n\|\leq t_n K$, where $K=\sup\{\|y\|: y\in\Delta\}$. Thus, in this case x belongs to $\text{pos}(\Delta)$. If $f(x)>0$, then $(y_n)_{n\in\mathbf{N}}$ converges to $x/f(x)$ in norm:

Let $\varepsilon>0$ be arbitrary. Then there exists $N\in\mathbf{N}$ such that $|t_n-f(x)|<\varepsilon f(x)/2K$ and $\|x-x_n\|<\varepsilon f(x)/2$, for all $n>N$. Thus, we have

$$\begin{aligned}\left\|\frac{x}{f(x)}-y_n\right\| &= \frac{1}{f(x)}\|x-f(x)y_n\| \\ &\leq \frac{1}{f(x)}(\|x-t_n y_n\|+\|t_n y_n-f(x)y_n\|) \\ &< \frac{1}{f(x)}\left(\frac{\varepsilon f(x)}{2}+\frac{\varepsilon f(x)}{2K}K\right) \\ &= \varepsilon\end{aligned}$$

for all $n>N$. Since Δ is closed, we conclude that $x/f(x)\in\Delta$. Hence, $x\in\text{pos}(\Delta)$. ■

Theorem 4.13. Let $(X, \|\cdot\|)$ be a real Banach space and suppose that Δ is a closed, bounded, convex subset of X such that (i) $X=\text{lin}(\Delta)$; and (ii) there exists $f\in X^*$ with $f(\Delta)=\{1\}$.

Then the Minkowski functional $\rho_{\text{acon}(\Delta)}$ is a norm on X, equivalent to $\|\cdot\|$.

Proof. Let the closed unit ball of X be denoted by X_1, and let $\bar{A}$ denote the closure of a set A in X. From the previous lemma it follows that $\text{pos}(\Delta)$ is closed. Moreover, since $\text{lin}(\Delta)=X$, $\text{pos}(\Delta)$ is generating. For each $c\in\mathbf{R}_+$ we set

$$P_c=\{x\in\text{pos}(\Delta): \|x\|\leq c\}$$

and

$$B_c = \mathrm{acon}(P_c)$$

Since pos(Δ) is generating,

$$X = \bigcup_{n \in \mathbf{N}} nB_c$$

holds for all $c > 0$. Now, let c be an arbitrary positive number. It then follows from the Baire category theorem that one of the sets $n\overline{B_c}$ has an interior point (Dunford and Schwartz, 1958). Since this set is convex and symmetrical about 0, it then also contains 0 as an interior point. Hence there exists $\gamma > 0$ such that $X_1 \subseteq \gamma \overline{B_c}$, and for all $x \in X$ we have

$$\frac{1}{\gamma} \rho_{\overline{B_c}} \leq \|x\| \leq c \rho_{\overline{B_c}}$$

(the inequality to the right being a consequence of $\overline{B_c} \subseteq cX_1$). This shows that for all $c > 0$ the Minkowski functional $\rho_{\overline{B_c}}$ is a norm on X, equivalent to $\|\cdot\|$. Since the triple $(X, \|\cdot\|, \mathrm{pos}(\Delta))$ satisfies all conditions of Theorem A.4, we conclude that the statement

$$\rho_{\overline{B_c}} = \rho_{B_c}$$

holds true for $c = 1$. However, $B_c = cB_1$ for all $c \in \mathbf{R}$, so that the statement holds true in general. Since Δ is bounded and there exists $f \in X^*$ with $f(\Delta) = \{1\}$, there exist constants $0 < a < b$ such that $a \leq \|x\| \leq b$ for all $x \in \Delta$. It is easily seen that $B_a \subseteq \mathrm{acon}(\Delta) \subseteq B_b$. For all $x \in X$ we thus have

$$\frac{1}{b} \|x\| \leq \rho_{\overline{B_b}}(x) = \rho_{B_b}(x) \leq \rho_{\mathrm{acon}(\Delta)}(x) \leq \rho_{B_a}(x) = \rho_{\overline{B_a}}(x) \leq \alpha \|x\|$$

where α is chosen such that $X_1 \subseteq \alpha \overline{B_a}$. This concludes the proof. ■

Corollary 4.14. Let $(X, \|\cdot\|)$ be a real Banach space and suppose that Δ is a closed, bounded, convex subset of X satisfying the conditions (i) and (ii) of the preceding theorem. Then every *CBT* $T: \Delta \to \Delta$ has a unique extension to a continuous linear transformation T' of X.

Proof. Let T be an arbitrary *CBT* of Δ. Then, by Theorem 4.2, T has a unique extension T' to a linear transformation of X. We have to show that T' is continuous: By the preceding theorem, there exist $a, b > 0$ such that

$$a\rho_{\mathrm{acon}(\Delta)}(x) < \|x\| < b\rho_{\mathrm{acon}(\Delta)}(x)$$

for all $x \in X$. Now, let x be an arbitrary element of X. Then

$$x \in (\|x\|/a)\ \mathrm{acon}(\Delta)$$

Since T' maps acon(Δ) into itself, we also have $T'(x) \in (\|x\|/a)$ acon(Δ). On the other hand, $y \in (\|x\|/a)$ acon(Δ) implies that

$$\|y\|/b < \rho_{\mathrm{acon}(\Delta)}(y) \leq \|x\|/a$$

Hence $\|T'(x)\| < (b/a)\|x\|$. ■

We now proceed to the proof of Theorem 4.10.

Proof of Theorem 4.10. Let ω be an arbitrary element of $\Omega_\sigma^s(\mathscr{D})$, and assume that $T: \Delta \to \Delta$ is a *CBT*. Let further g be an arbitrary element of X^*. Then, as a consequence of Corollary 4.14, $g \circ T'$ belongs to X^*, too. As in the proof of Theorem 3.4, we shall use the results on the Bochner integral listed in the Appendix. Again, we let Δ, $\mathscr{D}$, and an arbitrary element ω of $\Omega_\sigma^s(\mathscr{D})$ take the roles of S, Σ, and μ, respectively. Moreover, we set $X = \Delta$, $Y = \mathbf{R}$, $f = \mathrm{id}_\Delta$, and $L = g \circ T'$. Having the proof of Theorem 3.4 in mind, it is easy to see that thereby the conditions of Theorem A.3 are satisfied. Hence we have

$$\begin{aligned} g \circ T'(j'(\omega)) &= g \circ T' \int_\Delta x\ \omega(dx) \\ &= \int_\Delta g \circ T'(x)\ \omega(dx) \end{aligned}$$

As in the proof of Theorem 4.8, one then shows that

$$\begin{aligned} \int_\Delta g \circ T'(x)\ \omega(dx) &= \int_\Delta g(x)\ (\tilde{T}\omega)(dx) \\ &= j(\tilde{T}(\omega))(g) \\ &= g(j' \circ \tilde{T}(\omega)) \end{aligned}$$

This proves that $T \circ j'(\omega) = j' \circ \tilde{T}(\omega)$. ■

5. STATE SPACE TRANSFORMATIONS

In this section we apply the results of the previous sections to the state spaces of noncommutative measure theory and to their transformations.

Throughout the section let $(L, \leq, ')$ denote an orthomodular poset. In its most general definition, a *state space* over L is a nonempty convex subset Δ of $\Omega(L)$ (Fischer and Rüttimann, 1978; Mielnik, 1974).

Let Δ be a state space. Since Δ is convex, it is the base of a generating cone in $\text{lin}(\Delta)$. It is not hard to show that the Minkowski functional $\rho_{\text{acon}(\Delta)}$ on $\text{lin}(\Delta)$ provides a norm, which we shall denote by $\|\cdot\|_\Delta$. A norm that can be defined on $J(L)$ and all its subspaces is the *variation norm*:

$$\|\mu\|_v = \sup\{\mu(p) - \mu(p') : p \in L\}$$

Notice that $\|\mu\|_v \leq \|\mu\|_\Delta$ holds for all $\mu \in \text{lin}(\Delta)$. We also consider the topology τ of *pointwise convergence* on $J(L)$ and its subspaces [i.e., a net (μ_δ) of elements of $J(L)$ converges to $\mu \in J(L)$ in the topology τ if $\mu_\delta(p) \to \mu(p)$ for all $p \in L$]. Considering $J(L)$ as a subspace of $\mathbf{R}^L$, τ appears as the relative topology on $J(L)$ of the product topology on $\mathbf{R}^L$. It then follows from the Tychonov theorem that $\Omega(L)$ is τ-compact. Since every τ-open neighborhood U of 0 in $J(L)$ contains a set of the form $\{\mu \in J(L) : |\mu(p_i)| < 1/N, i = 1, \ldots, n\}$, where $p_1, \ldots, p_n$ are suitable elements of L and N is a suitable natural number, the set $\Omega(L)$ and all its subsets are τ-bounded. Moreover, since $\|\mu\|_\Delta \geq \|\mu\|_v$ for all $\mu \in \text{lin}(\Delta)$, the set Δ is bounded in both norms. Notice also that $\|\cdot\|_\Delta$-convergence implies $\|\cdot\|_v$-convergence and $\|\cdot\|_v$-convergence implies τ-convergence. For more details, see, e.g., Rüttimann (1979, 1985), Schindler (1986), and Zierler (1959).

Cone base transformations of a state space Δ will be referred to as *state space transformations*. Thus, all we require of a state space transformation is that it preserve mixtures (convex combinations) of states. Since the unit ball of the normed space $(\text{lin}(\Delta), \|\cdot\|_\Delta)$ coincides with the intersection of the sets $(1 + 1/n)\,\text{acon}(\Delta)$, $n \in \mathbf{N}$, the unique linear extension T' of a state space transformation T is $\|\cdot\|_\Delta$-continuous.

Theorem 5.1. Let Δ be a state space over L, and let $\mathcal{D}$ denote the σ-algebra on Δ generated by the τ-relatively open subsets of Δ. Suppose that $T: \Delta \to \Delta$ is a τ-continuous state space transformation. Then the diagram in Figure 3 is commutative. If, in addition, Δ is τ-closed, then the diagram in Figure 4 is commutative.

Proof. The first part of the above theorem is only a paraphrase of Theorem 4.8. Assume that Δ is τ-closed. Then, as a subset of the τ-compact set $\Omega(L)$, Δ is also τ-compact. Thus, Corollary 4.9 applies. ■

It is easy to see that in the special case when Δ equals $\Omega(L)$, resp. $\Omega_\sigma(L)$, resp. $\Omega_c(L)$, every homomorphism, resp. σ-homomorphism, resp. c-homomorphism, Φ from $(L, \leq, ')$ to itself induces a τ-continuous state space transformation T of $\Omega(L)$, resp. $\Omega_\sigma(L)$, resp. $\Omega_c(L)$, according to

$$T(\mu)(p) = \mu(\Phi(p))$$

for all $p \in L$.

If the state space transformation $T: \Delta \to \Delta$ under consideration is not τ-continuous, then it is often possible to apply Theorem 4.10 in the following form:

Theorem 5.2. Let Δ be a state space over L, and let $\mathrm{lin}(\Delta)$ be endowed with one of the norms $\|\cdot\| = \|\cdot\|_\Delta$ or $\|\cdot\| = \|\cdot\|_v$. Let $\mathscr{D}$ denote the σ-algebra on Δ generated by the $\|\cdot\|$-relatively open subsets of Δ. Assume that Δ' is $\|\cdot\|$-closed and $\mathrm{lin}(\Delta)$ is complete with respect to the norm $\|\cdot\|$. Then the diagram in Figure 5 is commutative for every state space transformation $T: \Delta \to \Delta$. If, in addition, $\mathrm{lin}(\Delta)$ is $\|\cdot\|$-separable, then the diagram in Figure 4 is commutative.

Proof. Let $\mathrm{lin}(\Delta)^*$ denote the dual of the Banach space $X = \mathrm{lin}(\Delta)$. We have shown that Δ is $\|\cdot\|$-bounded. Let f_1 be the linear functional on $\mathrm{lin}(\Delta)$ defined by $f_1(\mu) = \mu(1)$, $\mu \in \mathrm{lin}(\Delta)$. Clearly, f_1 is τ-continuous and thus also belongs to $\mathrm{lin}(\Delta)^*$. Since $f_1(\Delta) = \{1\}$, we see that X and Δ satisfy all conditions of Theorem 4.10 with respect to the norm $\|\cdot\|$. This proves the first part of the theorem. If, in addition, $\mathrm{lin}(\Delta)$ is $\|\cdot\|$-separable, then we apply Corollary 4.11 to prove the second part of the theorem. ■

If Δ is a nonempty *section* of $\Omega(L)$ [i.e., $\Delta = \mathrm{aff}(\Delta) \cap \Omega(L)$], and if Δ is *σ-convex* (i.e., for all sequences (x_n) in Δ and all sequences (t_n) in $\mathbf{R}_+$ with $\sum_{n=1}^{\infty} t_n = 1$, there exists $x \in \Delta$ with $x = \tau - \lim_{N\to\infty} \sum_{n=1}^{N} t_n x_n$], then $\mathrm{lin}(\Delta)$ is complete with respect to the norm $\|\cdot\|_\Delta$ (Rüttimann and Schindler, 1987).

Let Δ be any of the sets $\Omega(L)$, $\Omega_\sigma(L)$, or $\Omega_c(L)$. Then Δ is a σ-convex section of $\Omega(L)$ (Rüttimann and Schindler, 1987). Thus, $(\mathrm{lin}(\Delta), \|\cdot\|_\Delta)$ is a Banach space. Since $\Omega(L)$ is τ-closed and Δ equals $\Omega(L) \cap \mathrm{lin}(\Delta)$, it follows that Δ is $\|\cdot\|_\Delta$-closed. Hence we obtain the following theorem.

Theorem 5.3. Let Δ be any of the sets $\Omega(L)$, $\Omega_\sigma(L)$, or $\Omega_c(L)$, and assume that Δ is nonempty. Let $\mathscr{D}$ denote the σ-algebra on Δ generated by the $\|\cdot\|_\Delta$-relatively open subsets of Δ. Then the diagram in Figure 5 is commutative for every state space transformation $T: \Delta \to \Delta$. If, in addition, $\mathrm{lin}(\Delta)$ is $\|\cdot\|_\Delta$-separable, then the diagram in Figure 4 is commutative.

We conclude this section with an example.

Let H be a complex Hilbert space, and let L denote the set of all orthogonal projections on H. Then the order $P \le Q \Leftrightarrow P = PQ$ $(= QP)$ makes L into a complete lattice with id_H as the largest and the 0-projection as the smallest element. Together with the orthocomplementation $P' = \mathrm{id}_H - P$, the pair $(L, \le)$ forms a complete orthomodular lattice (Gudder, 1979). Let $\tilde{\Delta} = \Omega_c(L)$. As a consequence of the Gleason theorem, there exists a linear isomorphism Ψ from the real vector space $\mathrm{lin}(\tilde{\Delta})$ onto the real vector space

$\mathscr{T}_{sa}(H)$ of self-adjoint trace class operators on H, satisfying

$$\mu(P)=\operatorname{tr}(\Psi(\mu)P)$$

for all $P\in L$ and $\mu\in\operatorname{lin}(\tilde{\Delta})$. This isomorphism maps $\tilde{\Delta}$ onto the set $\mathscr{D}(H)$ of von Neumann density operators [i.e., positive elements of $\mathscr{T}_{sa}(H)$ with trace norm 1]. More precisely, the map Ψ is an isometrical isomorphism between the two Banach spaces $(\operatorname{lin}(\tilde{\Delta}), \|\cdot\|_{\tilde{\Delta}})$ and $(\mathscr{T}_{sa}(H), \|\cdot\|_{tr})$, where $\|\cdot\|_{tr}$ denotes the trace norm. For more details see Gleason (1957) and Rüttimann (1957).

Theorem 5.4. Let H be a complex Hilbert space. Let X denote the real vector space of self-adjoint trace class operators on H and let Δ stand for the convex set $\mathscr{D}(H)$ of von Neumann density operators on H. Let further $\mathscr{D}$ denote the σ-algebra on Δ generated by the $\|\cdot\|_{tr}$-relatively open subsets of Δ. Then Δ is the base of a generating cone in X and for every cone base transformation $T\colon\Delta\to\Delta$ the diagram in Figure 5 is commutative. If H is separable, then the diagram in Figure 4 is commutative.

Proof. Notice that $\mathscr{T}_{sa}(H)=\operatorname{lin}(\Delta)$ and $\|\cdot\|_{tr}=\|\cdot\|_{\Delta}$. Since Ψ is an isometrical isomorphism between the real Banach spaces $(\operatorname{lin}(\tilde{\Delta}), \|\cdot\|_{\tilde{\Delta}})$ and $(\mathscr{T}_{sa}(H), \|\cdot\|_{tr})$, mapping $\tilde{\Delta}$ onto Δ, the first part of the theorem follows from the first part of Theorem 5.3. If H is separable, then $\mathscr{T}_{sa}(H)$ contains a countable $\|\cdot\|_{tr}$-dense subset of operators of finite rank and the second part of Theorem 5.3 applies. ■

APPENDIX

This Appendix is meant to provide a short review of the basic results on the Bochner integral used in the paper. For a detailed treatment of these topics we refer to Dunford and Schwartz (1958) and Yosida (1966). For the sake of completeness we also include a result on base normed spaces used in Section 4.

Let (S, Σ, μ) be a positive measure space with $\mu(S)<\infty$, and let X be a Banach space. Let further $\mathscr{B}(X)$ denote the Borel σ-algebra on X (i.e., the σ-algebra generated by the open subsets of X). A Σ-$\mathscr{B}(X)$-measurable function $f\colon S\to X$ is called *μ-simple* if there exists a finite subset F of X such that $\mu(f^{-1}(F))=\mu(S)$. In this case the integral of f over S is given by $\sum_{x\in F} x\mu(f^{-1}(x))$.

A function $f\colon S\to X$ is said to be *μ-measurable* if there exists a sequence of μ-simple functions $h_n\colon S\to X$ converging to f in μ-measure. The function

f is called μ-integrable if, in addition,

$$\lim_{m,n\to\infty} \int_S |h_m(s) - h_n(s)| \, \mu(ds) = 0$$

The integral of f over S is then defined as the limit in X of the sequence $\int_S h_n(s) \, \mu(ds)$.

Lemma A.1. Let X and Y be Banach spaces and suppose that $f: S \to X$ and $g: S \to Y$ are both μ-measurable. If g is μ-integrable and $\|f(s)\| \leq \|g(s)\|$ holds μ-a.e., then f is μ-integrable.

For a proof see Dunford and Schwartz (1958), pp. 117ff.

Theorem A.2. A function $f: S \to X$ is μ-measurable if and only if (i) f is μ-essentially separably valued (i.e., there exists a μ-null set N such that $f(S-N)$ is separable in X); and (ii) for every $x^* \in X^*$, $x^* f$ is μ-measurable.

For a proof see Dunford and Schwartz (1958), pp. 149ff.

Theorem A.3. Let X and Y be Banach spaces. Assume that $f: S \to X$ is a μ-integrable function and that $L: X \to Y$ is a continuous linear operator. If $L \circ f$ is also μ-integrable, then $L \int_S f(s) \, \mu(ds) = \int_S L \circ f(s) \, \mu(ds)$.

For a proof see Dunford and Schwartz (1958), pp. 153ff.

Theorem A.4. Let $(X, \|\cdot\|)$ be a real Banach space with a closed generating cone P. Let P_1 denote the set $\{x \in P: \|x\| \leq 1\}$, and set $B = \operatorname{conv}(P_1 \cup -P_1)$. Then the Minkowski functional $\rho_B(\cdot)$ is a norm on X, equivalent to $\|\cdot\|$. Moreover, ρ_B coincides with $\|\cdot\|$ on P, and $\|\cdot\| - \operatorname{cl} B \subseteq tB$ holds for all $t > 1$. Thus, the two Minkowski functionals $\rho_{\|\cdot\| - \operatorname{cl} B}$ and ρ_B coincide.

An equivalent version of this theorem is proved in Asimov and Ellis (1980), p. 32.

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